Title	Vector Valued Pseudodifferential Operators and Their Applications (代数解析学とその応用)
Author(s)	TAIRA, KAZUAKI
Citation	数理解析研究所講究録 (1975), 226: 39-48
Issue Date	1975-02
URL	http://hdl.handle.net/2433/105380
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
226巻 1975年 39-48

Vector Valued Pseudodifferential Operators and their Applications

By Kazuaki TAIRA

Department of Mathematics, Tokyo Institute of Technology

§1. Introduction and the result. In this note we introduce the vector valued pseudodifferential operators where the vector space depends on a parameter (cf. Sjöstrand [3], §4) and take this opportunity to construct global parametrix-like operators for the following operator in R^n:

$$P(x,t,D_x,D_t) = D_t - it^k a(x,t,D_x,D_t) + b(x,t,D_x,D_t),$$

where $(x,t)\in R^n$ with $x\in R^{n-1}$, $t\in R$ and $k\in Z^+$ <u>is odd</u>, and $a(x,t,D_x,D_t)$ and $b(x,t,D_x,D_t)$ are properly supported classical pseudodifferential operators of order 1 and order 0 respectively, and the principal symbol $a_1(x,t,\xi,\tau)$ of $a(x,t,D_x,D_t)$ is positively homogeneous of degree 1 and

$$\text{(A)} \qquad \operatorname{Re} a_1(x,t,\xi,\tau) \neq 0$$

for $(x,t)\in R^n$, $(\xi,\tau)\neq(0,0)$, $\xi\in R^{n-1}$, $\tau\in R$ (cf. Sjöstrand [2]). Introducing the vector valued pseudodifferential operators, we can construct the parametrix-like operators in the above non-elliptic case completely analogous to the elliptic case.

Theorem. (cf. [2], Theorem 1.) <u>Assume that for all</u> ℓ, m $Z^+\cup\{0\}$ <u>and multiindices</u> α,β <u>there exists a constant</u> C =

$C(\alpha,\ell,\beta,m)$ such that

(B) $$\left|D_x^{\alpha}D_t^{\ell}D_{\xi}^{\beta}D_{\tau}^{m}a_1(x,0,\xi,\tau)\right| \leq C(1+|\xi|+|\tau|)^{1-|\beta|-m}$$

for $x\in R^{n-1}$, $(\xi,\tau)\neq(0,0)$.

I. (i) If $\operatorname{Re} a_1(x,t,\xi,\tau)>0$ for $(x,t)\in R^n$ and $(\xi,\tau)\neq(0,0)$, then there exist properly supported operators

$$\mathcal{P}_1=\begin{pmatrix}P\\ R^+\end{pmatrix}: \mathcal{D}'(R^n)\longrightarrow \begin{matrix}\mathcal{D}'(R^n)\\ \oplus\\ \mathcal{D}'(R^{n-1})\end{matrix}$$

$$\mathcal{G}_1=(G_1,G^+): \begin{matrix}\mathcal{D}'(R^n)\\ \oplus\\ \mathcal{D}'(R^{n-1})\end{matrix}\longrightarrow \mathcal{D}'(R^n)$$

such that $\mathcal{G}_1\cdot\mathcal{P}_1-I$ and $\mathcal{P}_1\cdot\mathcal{G}_1-I$ have C^∞ kernels.

(ii) If $\operatorname{Re} a_1(x,t,\xi,\tau)<0$ for $(x,t)\in R^n$ and $(\xi,\tau)\neq(0,0)$, then there exist properly supported operators

$$\mathcal{P}_2=(P,R^-): \begin{matrix}\mathcal{D}'(R^n)\\ \oplus\\ \mathcal{D}'(R^{n-1})\end{matrix}\longrightarrow \mathcal{D}'(R^n)$$

$$\mathcal{G}_2=\begin{pmatrix}G_2\\ G^-\end{pmatrix}: \mathcal{D}'(R^n)\longrightarrow \begin{matrix}\mathcal{D}'(R^n)\\ \oplus\\ \mathcal{D}'(R^{n-1})\end{matrix}$$

such that $\mathcal{G}_2\cdot\mathcal{P}_2-I$ and $\mathcal{P}_2\cdot\mathcal{G}_2-I$ have C^∞ kernels.

II. For all $s\in R$,

$$G_1,G_2:H_s^{loc}(R^n)\longrightarrow H^{loc}_{s+\frac{1}{1+k}}(R^n),$$

$$G^+:H_s^{loc}(R^{n-1})\longrightarrow H^{loc}_{s+\frac{1}{1+k}}(R^n),$$

$$G^-:H_s^{loc}(R^n) \longrightarrow H_s^{loc}(R^{n-1})$$

are continuous.

III. $WF'(G_1)$, $WF'(G_2) \subset \{((x,t,\xi,\tau),(x,t,\xi,\tau)) \in (T^*(R^n)\setminus 0) \times (T^*(R^n)\setminus 0)\}$;

$WF'(R^-)$, $WF'(G^+) \subset \{((x,0,\xi,0),(x,\xi)) \in (T^*(R^n)\setminus 0) \times (T^*(R^{n-1})\setminus 0)\}$;

$WF'(R^+)$, $WF'(G^-) \subset \{((x,\xi),(x,0,\xi,0))\ (T^*(R^{n-1})\setminus 0) \times (T^*(R^n)\setminus 0)\}$.

§2. Vector valued pseudodifferential operators. (cf. Treves [4], Theorem 4.1.) Let H_1 and H_2 be complex Hilbert spaces and let $\mathcal{L}(H_1,H_2)$ be the Banach space of bounded linear operators $H_1 \longrightarrow H_2$. We define $S^m(R^n \times R^n;\ H_1,H_2)$ as the space of C^∞ functions $p(x,\xi)$ on $R^n \times R^n$ with values in $\mathcal{L}(H_1,H_2)$ such that for all $K \subset\subset R^n$ and multiindices α,β there exists a constant $C = C(\alpha,\beta,K)$ such that

$$\| D_x^\alpha D_\xi^\beta p(x,\xi) \|_{\mathcal{L}(H_1,H_2)} \le C(1 + |\xi|)^{m-|\beta|}$$

for all $(x,\xi) \in K \times R^n$. With such symbols we define $L^m(R^n;\ H_1,H_2)$ to be the space of pseudodifferential operators $P(x,D_x): C_0^\infty(R^n;\ H_1) \longrightarrow C^\infty(R^n;\ H_2)$.

We shall consider the case that H_1 or H_2 is equal to the space $D_\xi^k(R)$ with $k \in Z^+$, $\xi \in R^{n-1}$, which is a subspace of $H^1(R)$, given by the norm:

$$\|u\|^2_{D_\xi^k} = (1 + |\xi|)^{\frac{2}{1+k}} \int_{-\infty}^{\infty} |u(t)|^2 dt \ +$$

(continued)

$$(1+|\xi|)^2 \int_{-\infty}^{\infty} t^{2k} \, |u(t)|^2 dt \; + \int_{-\infty}^{\infty} |D_t u(t)|^2 dt.$$

(cf. [3], §4.)

In this case the norm $\| \;\; \|_{\mathcal{L}(H_1,H_2)}$ <u>depends on</u> $\xi \in R^{n-1}$, but all the calculus for scalar operators (cf. Hörmander [1]) extends to the vector valued case, in particular we have the usual composition formula and the results about H_s-continuity because we have the inequality:

$$\|u\|_{D^k} \leq \|u\|_{D^k_\xi} \leq (1+|\xi|) \, \|u\|_{D^k}$$

and hence

$$L^m(R^{n-1};\, H_1, D^k_\xi(R)) \subset L^m(R^{n-1};\, H_1, D^k(R)),$$

$$L^m(R^{n-1};\, D^k_\xi(R), H_2) \subset L^{m+1}(R^{n-1};\, D^k(R), H_2),$$

where $D^k(R)$ is the space $D^k_\xi(R)$ with $\xi = 0$.

We define $T^m(R^n)$ to be the space of "pseudodifferential operators" $a(x,t,D_x): C_0^\infty(R^n) \longrightarrow C^\infty(R^n)$ where $a(x,t,\xi) \in S^m(R^n \times R^{n-1})$ (see [2], Appendix).

Under the assumptions (A), (B), we can reduce the proof of Theorem to the proof of

Proposition. (cf. [2], Proposition 3.6.) <u>Let</u> $L(x,t,D_x,D_t) = D_t - it^k\, r(x,t,D_x) + s(x,t,D_x)$, <u>where</u> $r(x,t,D_x) \in T^1(R^n)$ (<u>resp</u>. $s(x,t,D_x) \in T^0(R^n)$) <u>is properly supported and its symbol</u> $r(x,t,\xi)$ <u>is positively homogeneous of degree</u> 1 <u>and</u> $r(x,t,\xi)$ (<u>resp</u>.

$s(x,t,\xi))$ *is equal to* $r(x,0,\xi)$ (*resp.* $s(x,0,\xi)$) *when* $|t| \geq C$ *for some constant* $C > 0$ *and* $\mathrm{Re}\, r(x,t,\xi) \neq 0$ *for* $(x,t) \in R^n$ *and* $\xi \neq 0$.

(i) *If* $\mathrm{Re}\, r(x,t,\xi) > 0$ *for* $(x,t) \in R^n$ *and* $\xi \neq 0$, *then there exist properly supported operators*

$$\mathcal{L}_1(x,D_x) = \begin{pmatrix} L(x,t,D_x,D_t) \\ \\ R^+(x,D_x) \end{pmatrix} \in L^0(R^{n-1};\ D^k_\xi(R), L^2(R) \oplus C),$$

$$\mathcal{E}_1(x,D_x) = (E_1(x,D_x),\ E^+(x,D_x)) \in L^0(R^{n-1};\ L^2(R) \oplus C, D^k_\xi(R))$$

such that

$$\mathcal{L}_1(x,D_x) \cdot \mathcal{E}_1(x,D_x) \equiv I \bmod L^{-\infty}(R^{n-1};\ L^2(R) \oplus C, L^2(R) \oplus C),$$

$$\mathcal{E}_1(x,D_x) \cdot \mathcal{L}_1(x,D_x) \equiv I \bmod L^{-\infty}(R^{n-1};\ D^k_\xi(R), D^k_\xi(R)).$$

(ii) *If* $\mathrm{Re}\, r(x,t,\xi) < 0$ *for* $(x,t) \in R^n$ *and* $\xi \neq 0$, *then there exist properly supported operators*

$$\mathcal{L}_2(x,D_x) = (L(x,t,D_x,D_t),\ R^-(x,D_x)) \in L^0(R^{n-1};\ D^k_\xi(R) \oplus C, L^2(R)),$$

$$\mathcal{E}_2(x,D_x) = \begin{pmatrix} E_2(x,D_x) \\ \\ E^-(x,D_x) \end{pmatrix} \in L^0(R^{n-1};\ L^2(R), D^k_\xi(R) \oplus C)$$

such that

$$\mathcal{L}_2(x,D_x) \cdot \mathcal{E}_2(x,D_x) \equiv I \bmod L^{-\infty}(R^{n-1};\ L^2(R), L^2(R)),$$

$$\mathcal{E}_2(x,D_x) \cdot \mathcal{L}_2(x,D_x) \equiv I \bmod L^{-\infty}(R^{n-1};\ D^k_\xi(R) \oplus C, D^k_\xi(R) \oplus C).$$

§3. Sketch of the proof of Proposition.

Lemma 1. Let $L(x,\xi) = L_0(x,\xi) + L_1(x,\xi)$, where $L_0(x,\xi) = D_t - it^k\, r(x,t,\xi)$, $L_1(x,\xi) = s(x,t,\xi)$. Then we have

$$L_0(x,\xi) \in S^0(R^{n-1}\times R^{n-1};\ D^k_\xi(R), L^2(R)),$$

$$L_1(x,\xi) \in S^{-\frac{1}{1+k}}(R^{n-1}\times R^{n-1};\ D^k_\xi(R), L^2(R)).$$

In particular, we can regard $L(x,t,D_x,D_t)$ as an element of $L^0(R^{n-1};\ D^k_\xi(R), L^2(R))$.

The next lemma is the essential step in our proof of Proposition.

Lemma 2. Let $B(x,t,s,\xi) = -\int_s^t \theta^k\, r(x,\theta,\xi)d\theta.$

(i) When $\operatorname{Re} r(x,t,\xi) > 0$ for $(x,t)\in R^n$, $\xi \neq 0$, we define the kernel $K_1(x,t,s,\xi)$ by

$$K_1(x,t,s,\xi) = \begin{cases} i \exp\left[B(x,t,s,\xi)\right] & 0 \le s \le t, \\ -i \exp\left[B(x,t,s,\xi)\right] & t \le s \le 0, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) When $\operatorname{Re} r(x,t,\xi) < 0$ for $(x,t)\in R^n$, $\xi\neq 0$, we define the kernel $K_2(x,t,s,\xi)$ by

$$K_2(x,t,s,\xi) = \begin{cases} -i \exp\left[B(x,t,s,\xi)\right] & 0 \le t \le s, \\ i \exp\left[B(x,t,s,\xi)\right] & s \le t \le 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $K_j(x,t,s,\xi)$ $(j=1,2)$ we have the following estimates:

$$(1)\quad \sup_t \int_{-\infty}^{\infty} |K_j(x,t,s,\xi)|\,ds = O(|\xi|^{-\frac{1}{1+k}}),\quad \xi \longrightarrow \infty,$$

$$\sup_s \int_{-\infty}^{\infty} |K_j(x,t,s,\xi)|\,dt = O(|\xi|^{-\frac{1}{1+k}}),\quad \xi \longrightarrow \infty,$$

<u>uniformly when</u> x <u>belongs to any compact subset of</u> R^{n-1}.

$$(2)\quad \sup_t \int_{-\infty}^{\infty} |t|^k\,|K_j(x,t,s,\xi)|\,ds = O(|\xi|^{-1}),\quad \xi \longrightarrow \infty,$$

$$\sup_s \int_{-\infty}^{\infty} |t|^k\,|K_j(x,t,s,\xi)|\,dt = O(|\xi|^{-1}),\quad \xi \longrightarrow \infty,$$

<u>uniformly when</u> x <u>belongs to any compact subset of</u> R^{n-1}.

Lemma 2 follows from the following two facts (cf. Treves [4], Lemma C.1):

(a) There exists a constant $C_1 > 0$ such that

$$|t - s|^{k+1} \leq C_1\,|t^{k+1} - s^{k+1}|$$

for all $t\cdot s \geq 0$ when $k \in Z^+$ is odd.

(b) There exists a constant $C_2 > 0$ such that

$$|t|^k\,|t - s| \leq C_2\,|t^{k+1} - s^{k+1}|$$

for all $t\cdot s \geq 0$ when $k \in Z^+$ is odd.

Combining Corollary in [4], p. 94 and Lemma 2, we can prove

Lemma 3. <u>Let</u> Re $r(x,t,\xi) > 0$. <u>We define for</u> $|\xi| \geq 1$

$$R^+(x,\xi) : D^k_\xi(R) \longrightarrow C,$$

$$E_0^+(x,\xi): C \longrightarrow D_\xi^k(R),$$

$$E_{10}(x,\xi): L^2(R) \longrightarrow D_\xi^k(R),$$

by

$$R^+(x,\xi)u = |\xi|^{\frac{1}{1+k}} \int_{-\infty}^{\infty} u(t)\,\overline{\varphi(x,\xi,t)}\,dt,$$

$$E_0^+(x,\xi)z = |\xi|^{-\frac{1}{1+k}} \varphi(x,\xi,t)z,$$

$$E_{10}(x,\xi)f = \int_{-\infty}^{\infty} K_1(x,t,s,\xi)f(s)ds - E_0^+R^+K_1f,$$

respectively, where

$$\varphi(x,\xi,t) = \exp\left[-\int_0^t \theta^k r(x,\theta,\xi)d\theta\right] \Big/ \left(\int_{-\infty}^{\infty}\exp\left[-2\int_0^t \theta^k \operatorname{Re} r(x,\theta,\xi)d\theta\right]dt\right)^{\frac{1}{2}}.$$

Then, after having been suitably modified for small ξ,

$$\mathcal{L}_{10}(x,\xi) = \begin{pmatrix} L_0(x,\xi) \\ R^+(x,\xi) \end{pmatrix} \in S^0(R^{n-1}\times R^{n-1};\ D_\xi^k(R), L^2(R)\oplus C),$$

$$\mathcal{E}_{10}(x,\xi) = (E_{10}(x,\xi),\ E_0^+(x,\xi)) \in S^0(R^{n-1}\times R^{n-1};\ L^2(R)\oplus C, D_\xi^k(R))$$

and $\mathcal{E}_{10}(x,\xi)$ is the inverse of $\mathcal{L}_{10}(x,\xi)$ for $|\xi|\geq 1$.

Lemma 4. Let $\operatorname{Re} r(x,t,\xi) < 0$. We define for $|\xi|\geq 1$

$$R^-(x,\xi): C \longrightarrow L^2(R),$$

$$E_0^-(x,\xi): L^2(R) \longrightarrow C,$$

$$E_{2o}(x,\xi): L^2(R) \longrightarrow D^k_\xi(R),$$

by

$$R^-(x,\xi)z = \psi(x,\xi,t)z,$$

$$E_o^-(x,\xi)f = \int_{-\infty}^{\infty} f(t)\overline{\psi(x,\xi,t)}dt,$$

$$E_{2o}(x,\xi)f = \int_{-\infty}^{\infty} K_2(x,t,s,\xi)\ (f(s) - R^-E_o^-f(s))ds,$$

respectively, where

$$\psi(x,\xi,t) = \exp\left[\int_o^t \theta^k \overline{r(x,\theta,\xi)}d\theta\right] \Big/ \left(\int_{-\infty}^{\infty}\exp\left[2\int_o^t \theta^k \mathrm{Re}\ \overline{r(x,\theta,\xi)}d\theta\right]dt\right)^{\frac{1}{2}}.$$

Then, after having been suitably modified for small ξ,

$$\mathcal{L}_{2o}(x,\xi) = (L_o(x,\xi),\ R^-(x,\xi)) \in S^o(R^{n-1}\times R^{n-1};\ D^k_\xi(R)\oplus C, L^2(R)),$$

$$\mathcal{E}_{2o}(x,\xi) = \begin{pmatrix} E_{2o}(x,\xi) \\ \\ E_o^-(x,\xi) \end{pmatrix} \in S^o(R^{n-1}\times R^{n-1};\ L^2(R), D^k_\xi(R)\oplus C)$$

and $\mathcal{E}_{2o}(x,\xi)$ is the inverse of $\mathcal{L}_{2o}(x,\xi)$ for $|\xi|\geq 1$.

By Lemma 3 and Lemma 4 the construction of $\mathcal{E}_j(x,D_x)$ $(j=1,2)$ in Proposition is formally the same as the construction of a parametrix of an elliptic operator in the scalar case.

References

[1] Hörmander, L.: Pseudodifferential operators and hypoelliptic equations, Amer. Math. Soc. Symp. on Singular Integral Operators, 138-183 (1966).

[2] Sjöstrand, J.: Operators of principal type with interior boundary conditions, Acta Math., 130, 1-51 (1973).

[3] Sjöstrand, J.: Parametrices for pseudodifferential operators with multiple characteristics (to appear).

[4] Treves, F.: A new method of proof of the subelliptic estimates, Comm. Pure Appl. Math., 24, 71-115 (1971).